

Corresponding constant mean curvature surfaces in hyperbolic and Euclidean 3-spaces

Wayne Rossman, Magdalena Toda

Dedicated to the memory of Hongyou Wu

Abstract

We make observations about constant mean curvature surfaces in Euclidean 3-space and their dual surfaces, and the resulting pairs of surfaces in hyperbolic 3-space under the Lawson correspondence.

Mathematics Subject Classification: 53A10, 58E20

Key Words: constant mean curvature surface, Euclidean 3-space, hyperbolic 3-space, r-unitary frame.

It is well-known that any constant mean curvature (CMC) surface in Euclidean 3-space R^3 can be parameterized in conformal curvature line coordinates away from umbilics, and such a parameterization is called *isothermic*. A characteristic property of an isothermic immersion is the (local) existence of a *dual* surface f^d (via Christoffel transform, see [4] and [8]). Like in [8], we say that two surfaces in Euclidean 3-space form a Christoffel pair if they induce conformally equivalent metrics and have parallel tangent planes with opposite orientations. The Christoffel transform is an involution, and each of the surfaces of a Christoffel pair is called a Christoffel transform, or dual, of the other. The dual (Christoffel transform) is unique up to homothety and translation.

Let Σ be a simply-connected Riemann surface and let $f : \Sigma \rightarrow R^3$ be an isothermic constant mean curvature (CMC) H immersion that is not the round sphere. Assume $H \neq 0$, so the surface is not minimal, and let $N : \Sigma \rightarrow S^2$ (S^2 is the round sphere of radius 1 centered at the origin in R^3) be the unit normal vector to f . An appropriately scaled and positioned Christoffel transform of f will be its parallel constant mean curvature surface $f^d = f + H^{-1}N : \Sigma \rightarrow R^3$ with normal $N^d = -N$ (see, for example, [8]), which has the same constant mean curvature as f itself, i.e. $H_d = H$. It is well known that this Christoffel transform is also a Darboux transform; in fact, it was shown in [8] that the Christoffel transform f^d of an isothermic surface f is also a Darboux transform of f if and only if f has non-zero constant mean curvature H .

Taking the isothermic CMC immersions f and f^d as above, we can rescale the complex coordinate z of Σ so that f and f^d have induced conformal metrics that are inverse to each other, and we then call z a normalized isothermic coordinate. We find that

$$H = H_d = 2Q = 2Q_d ,$$

where Q and Q_d are the Hopf differential functions of f and f^d , respectively. The metric, Hopf differential function and mean curvature of f are

$$I = e^{2u}(dx^2 + dy^2) , \quad Q = \langle f_{zz}, N \rangle , \quad H = 2e^{-2u} \langle f_{z\bar{z}}, N \rangle ,$$

where $\langle \cdot, \cdot \rangle$ is the bilinear extension to complex 3-space C^3 of the usual Euclidean metric for R^3 . Then f^d has the corresponding data

$$I_d = e^{-2u}(dx^2 + dy^2) , \quad Q_d = \langle f_{zz}^d, N^d \rangle = Q , \quad H_d = 2e^{2u} \langle f_{z\bar{z}}^d, N^d \rangle = H .$$

Let the hyperbolic 3-space H^3 be given as an isometric submanifold of Minkowski 4-space $R^{3,1}$ (with $(+++-)$ metric) via $H^3 = \{(x_1, x_2, x_3, x_0) \in R^{3,1} \mid x_0 > 0, x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1\}$, which can also be written in the Hermitean matrix model as

$$H^3 = \left\{ \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \middle| x_0 > 0, x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1 \right\} = \{F\bar{F}^t \mid F \in \text{SL}_2C\} .$$

With respect to the Hermitean matrix model, points in $R^{3,1}$ can be given as

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

(where $\det X$ is not necessarily 1), and then the metric for $R^{3,1}$ is

$$\langle X, Y \rangle = -\frac{1}{2} \text{tr} \left(X \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} Y^t \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right)$$

for $X, Y \in R^{3,1}$.

We will be considering homotheties sf , resp. sf^d , of the immersion f , resp. f^d , for $s \in R \setminus \{0\}$. Then, the Lawson correspondent f_1 , resp. f_1^d , in H^3 , of sf , resp. sf^d , has the geometric data

$$(0.1) \quad I_1 = s^2 e^{2u}(dx^2 + dy^2) , \quad Q_1 = sQ , \quad H_1 = \sqrt{(s^{-1}H)^2 + 1} , \quad \text{resp.}$$

$$(0.2) \quad I_{d,1} = s^2 e^{-2u}(dx^2 + dy^2) , \quad Q_{d,1} = sQ , \quad H_{d,1} = \sqrt{(s^{-1}H)^2 + 1} .$$

Before stating our result, we note that the extended r -unitary frame for f can be given by a solution F to the following Lax system (see [2], [3], [6], [12], [13])

$$F_z = FU, \quad F_{\bar{z}} = FV, \\ U = \frac{1}{2} \begin{pmatrix} -u_z & 2e^{-u}\lambda^{-1}Q \\ -He^u & u_z \end{pmatrix}, \quad V = \frac{1}{2} \begin{pmatrix} u_{\bar{z}} & He^u \\ -2e^{-u}\lambda Q & -u_{\bar{z}} \end{pmatrix},$$

whose compatibility condition is the Gauss equation

$$4u_{z\bar{z}} - 4Q^2e^{-2u} + H^2e^{2u} = 0.$$

For the Euclidean immersion f , the parameter λ belongs to the unit circle S^1 , and then the frame F belongs to the SU_2 -valued loop group. The actual frame of f is represented by $F|_{\lambda=1}$. However, to create an r -unitary frame, we fix r as some positive real number less than one, and λ is taken to be in the open annular domain in the complex plane between the circles of radius r and radius $1/r$ centered at the origin. For more details on r -unitary frames, see [12]. Let D be the diagonal matrix

$$D = \begin{pmatrix} 1/\sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}.$$

Now we can state our result:

Theorem 1. *Let f be a CMC immersion in R^3 as above, without umbilic points and with normalized isothermic coordinate z , and with extended r -unitary frame F . Choose a value of λ so that $r < \lambda < 1$. Then the two surfaces $F\bar{F}^t$ and $FD\bar{F}\bar{D}^t$ in H^3 , each evaluated at that value of λ , are both isothermic CMC immersions in H^3 , and the following hold:*

1. $F\bar{F}^t$ is the Lawson correspondent to a homothety of f^d ;
2. $FD\bar{F}\bar{D}^t$ is the Lawson correspondent to a homothety of f ;
3. $FD\bar{F}\bar{D}^t$ is an equidistant (parallel) surface to $F\bar{F}^t$, with mean curvature opposite in sign to that of $F\bar{F}^t$.

Proof. The metric, Hopf differential function and (hyperbolic) mean curvature of $F\bar{F}^t$, resp. $FD\bar{F}\bar{D}^t$, are

$$(0.3) \quad Q^2e^{-2u}(\lambda - \lambda^{-1})^2(dx^2 + dy^2), \quad \frac{1}{2}QH(\lambda^{-1} - \lambda), \quad \frac{\lambda^{-1} + \lambda}{\lambda^{-1} - \lambda}, \quad \text{resp.}$$

$$(0.4) \quad Q^2e^{2u}(\lambda - \lambda^{-1})^2(dx^2 + dy^2), \quad \frac{1}{2}QH(\lambda - \lambda^{-1}), \quad \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}}.$$

These surfaces are isothermic CMC immersions in H^3 .

The metric, Hopf differential function and mean curvature of the Lawson correspondent of the homothety sf^d of f^d , resp. sf of f , are as in (0.2), resp. as in (0.1), and that data will be equal to the data (0.3) for $F\bar{F}^t$, resp. the data (0.4) for $FD\overline{F\bar{D}}^t$, if $s = \frac{1}{2}H(\lambda^{-1} - \lambda)$, resp. $s = \frac{1}{2}H(\lambda - \lambda^{-1})$, proving the first and second itemized statements of the theorem, as well as the second half of the third item.

Write $\lambda = e^q$ for some value $q < 0$. Noting that the normal vector to $(F\bar{F}^t)|_{\lambda=e^q}$ is

$$\hat{N} = \left(F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t \right) \Big|_{\lambda=e^q},$$

we have that $(FD\overline{F\bar{D}}^t)|_{\lambda=e^q} = (\cosh q)(F\bar{F}^t)|_{\lambda=e^q} - (\sinh q)\hat{N}$. Thus the distance between the surfaces $F\bar{F}^t$ and $FD\overline{F\bar{D}}^t$ is $-q$, completing the proof of the third item. \square

Remark 1. The surface $F\bar{F}^t$ was used in [12] to construct CMC trinoids in H^3 , thus those trinoids in H^3 are not Lawson correspondents to the CMC trinoids in R^3 with frame F , but rather to the duals of those CMC trinoids in R^3 . This shift to the dual surface in order to close periods on non-simply-connected CMC surfaces in H^3 occurred in [11] as well.

Remark 2. One simple example to which the theorem can be applied is round cylinders in H^3 , given by a round cylinder in R^3 with data

$$u = 0, \quad H = \frac{1}{2}, \quad Q = \frac{1}{4}$$

and

$$F = \begin{pmatrix} \cosh \gamma & (\sqrt{\lambda})^{-1} \sinh \gamma \\ \sqrt{\lambda} \sinh \gamma & \cosh \gamma \end{pmatrix}, \quad \gamma = \frac{1}{4}i \left(\frac{z}{\sqrt{\lambda}} + \sqrt{\lambda} \bar{z} \right).$$

Acknowledgement. The authors thank Udo Hertrich-Jeromin for noticing an error in their initial computations. They also note that Udo Hertrich-Jeromin found a different proof of the result here, using linear conserved quantities instead of unitary frames.

References

- [1] R. Aiyama, K. Akutagawa, *Kenmotsu-Bryant type representation formulas for constant mean curvature surfaces in $H^3(-c^2)$ and $S_1^3(c^2)$* , Annals of Global Analysis and Geometry **17**, 1998, 49-75.
- [2] R. Aiyama, K. Akutagawa, *Representation formulas for surfaces in $H^3(-c^2)$ and harmonic maps arising from CMC surfaces*, Harmonic Morphisms, Harmonic Maps, and

Related Topics, Chapman Hall/CRC Research Notes in Mathematics, **413**, 2000, 275-285.

- [3] A. I. Bobenko, *Constant mean curvature surfaces and integrable equations*, Russian Math. Surveys, **46:4**, 1991, 1-45.
- [4] E. Christoffel, *Ueber einige allgemeine Eigenschaften der Minimumsflächen*, Crelle's J, **67**, 1867, 218-228.
- [5] J. Dorfmeister, J. Inoguchi, M. Toda, *Weierstrass-type Representation of Timelike Surfaces with Constant Mean Curvature in Minkowski 3-Space*, Differential Geometry and Integrable Systems, Contemporary Mathematics AMS, vol. 308 (2002), pp 77-100.
- [6] J. Dorfmeister, F. Pedit, H. Wu, *Weierstrass-type representation of harmonic maps into symmetric spaces*, Comm. in Analysis and Geometry **6**, 1998, 633-668.
- [7] A. Fujioka, *Harmonic maps and associated maps from simply connected Riemann surfaces into 3-dimensional space forms*, Tôhoku Math. J., **47**, 1995, 431-439.
- [8] U. Hertrich-Jeromin, F. Pedit, *Remarks on the Darboux Transform of isothermic surfaces*; Doc. Math J. DMV 2 (1997), 313-333.
- [9] J. Inoguchi, M. Toda, *Timelike Minimal Surfaces via Loop Groups*, Acta Applicandae Mathematicae, vol. 83, no. 1-2, (2004) , pp 313-355.
- [10] C. Qing, C. Yi, *Spectral transformation of constant mean curvature surfaces in H^3 and Weierstrass representation*, Science in China, A, **45:8**, 2002, 1066-1075.
- [11] W. Rossman, M. Umehara, K. Yamada, *Irreducible constant mean curvature 1 surfaces in hyperbolic space with positive genus*, Tôhoku Math. J., **49**, 1997, 449-484.
- [12] N. Schmitt, M. Kilian, S-P. Kobayashi, W. Rossman, *Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimesnional space forms*, J. London Math. Soc., London Mathematical Society, 2007.
- [13] M. Toda, *Immersions of Constant Mean Curvature in Hyperbolic Space*, Differential Geometry, Dynamical Systems, vol. 7, no.1, (2005), pp 111-126.
- [14] M. Toda, *Initial Value Problems of the Sine-Gordon Equation and Geometric Solutions*, Annals of Global Analysis and Geometry, vol. 27, no. 3, (2005), pp 257-271.

Magdalena Toda,
 Department of Mathematics and Statistics,
 Texas Tech University,
 Lubbock, Texas 79409-1042,

U.S.A.

magda.toda@ttu.edu

Wayne Rossman,

Department of Mathematics, Faculty of Science,

Kobe University,

Rokko, Kobe 657-8501,

Japan

wayne@math.kobe-u.ac.jp